0-52 流体運動方程式の解法における境界層の役割

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A Role of Boundary Layer in a Solution of Equations of Fluid Motion

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SYNOPSIS

The author has observed that Prandtl's boundary layer theory has a defect due to the fact that Navier-Stokes' equation are derived from the assumpsion on smallness of rate of deformation. Therefore, the author generalizes equations of the fluid motion, using most general relations between the stress and the rate of deformation and reduces them into zero-dimensional forms in order to show the effects of Reynold's number and Mach's number.

Putting Mach's number equal to some power of the depth of boundary layer y_1, expanding the velocity components, the pressure and coefficients μ_1 and μ_2 into power series of y_1, and comparing coefficients of the same power of y_1, the author gets a system of ditterential equations which can be solved if either of the velocity components is given.

From the results obtained, the author can explain the effect of compressibility by increasing Mach's number and the mechanism of generation of turbulent flow.

1. 緒　　言

Prandtl は粘性流体の運動では粘性の影響は物体表面の薄い境界層内に限るとし境界層の厚さの小さいことを利用して境界層流体運動方程式を導き出した。[1] 実験的には境界層の境界を画然と決定出来るのにニュートン流体を基礎とするナビヤ・ストークスの流体運動方程式では境界層の境界条件を満すことが出来ない。そこで境界層流体運動方程式は境界層の70～80%の範囲で成立しその外では漸近的に無限遠における値に近づくものとして取扱われている。層流以外の流に対しては大部分が境界層流体運動方程式を利用せず他の方法で求めた方程式を使用して問題を解いているものが多い。[2] 流体を層流，乱流，非圧縮性流体，圧縮性流体などに区別して取扱つているがその間の関係について判然としないところがある。

これらのことから微小変位に対するフックの法則に基くナビア・ストークスの流体運動方程式の適用範囲に疑問を持つようになつた。

そこで一般化した応力と歪率との関係から流体運動方程式を導き出した。レイノルズ数，マッハ数の影響を考慮するためこれを零ヂメンションの方程式に直し式中にレイノルズ数，マッハ数が明かに入つてくるようにした。マッハ数を境界層の厚さのベキ数で表わし，いろいろのマッハ数に対し速度，圧力，応力と歪率との関係を求めた。その結果音速以下の流に対し，マッハ数の影響に考慮を払うだけで適当に仮定した境界層内の一つの速度成分の速度分布，一般化した流体運動方程式を使用して問題を解くことが出来るようになつた。音速以上の流に対しても同様に取扱うことが出来る。これにより圧縮性の影響，乱流の成因を明かにすることが出来た。既に報告した報告中[3] の数値の示すようにニュートン流体を実際の粘性流体の近似と見做し得る範囲は非常に小さいマッハ数の間に限られる。

2. 流体運動方程式の一般化

xy 平面上の 2 次元流の場合を考える。t を時間，ρ を密度，u, v を x, y 方向の分速度，F_x, F_y を x, y 方向の分力，$\tau_{xx}, \tau_{yx}, \tau_{xy}, \tau_{yy}$ を応力とすれば連続および流体運動方程式は

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

$$\frac{d}{dt}(\rho u)=F_x+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y},$$

$$\frac{d}{dt}(\rho v)=F_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}$$

で表わされる。

今 μ を粘性係数，$\mu_1=\mu_1(x,y,t)$, $\mu_2=\mu_2(x,y,t)$ とし応力と歪率との関係を

$$\frac{\tau_{xx1}}{\mu\frac{\partial u}{\partial x}}=\frac{\tau_{yy1}}{\mu\frac{\partial v}{\partial y}}=\frac{\tau_{xy}}{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)}=\frac{\tau_{yx}}{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)}=\mu_1,$$

$$\frac{\tau_{xx2}}{\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}=\frac{\tau_{yy2}}{\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}=\mu_2,$$

$$\tau_{xx}=-p+2\tau_{xx1}+\tau_{xx2}=-p+\mu\left\{2\mu_1\frac{\partial u}{\partial x}+\mu_2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\},$$

$$\tau_{yy}=-p+2\tau_{yy1}+\tau_{yy2}=-p+\mu\left\{2\mu_1\frac{\partial v}{\partial y}+\mu_2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\},$$

$$\tau_{xy}=\tau_{yx}=\mu\mu_1\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

のように一般化する。特に $\mu_1=1$, $\mu_2=-2/3$ のときはニュートン流体の場合に対応する。a を音速とし

$$\frac{\partial\rho}{\partial p}=1/a^2,\quad \frac{\partial\rho}{\partial t}=\frac{\partial\rho}{\partial p}\frac{\partial p}{\partial t}=\frac{\partial p}{\partial t}/a^2,\quad \frac{\partial\rho}{\partial x}=\frac{\partial p}{\partial x}/a^2,\quad \frac{\partial\rho}{\partial y}=\frac{\partial p}{\partial y}/a^2$$

であることを考慮して上の関係を流体運動方程式に代入すれば

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=-\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)/a^2,$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+u\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial p}\right)/a^2+\frac{\partial p}{\partial x}=F_x+\mu\left\{\mu_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(\mu_1+\mu_2)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial\mu_1}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\mu_1}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial\mu_2}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\},$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+v\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)/a^2+\frac{\partial p}{\partial y}=F_y+\mu\left\{\mu_1\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+(\mu_1+\mu_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial\mu_1}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial\mu_1}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial\mu_2}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}$$

となる。

ρ_b, p_b を境界層の境界における密度および圧力，V を物体運動の速さ，L を物体のある特定の長さとし

$x'=x/L$, $y'=y/L$, $t'=Vt/L$, $u'=u/V$, $v'=v/V$, $p'=p/(\rho V^2)$, $p_b'=p_b/(\rho_b V^2)$, $\bar{p}=p-p_b$, $\bar{p}'=\bar{p}/(\rho V^2)$, $F_x'=F_x/(\rho V^2 L)$, $F_y'=F_y/(\rho V^2 L)$, $R=\rho VL/\mu$, $M=V/a$, $M_b=V/a_b$

と置き上の方程式を零ディメンション化する。

$$\frac{\partial p}{\partial t}=\rho V^2\frac{\partial p'}{\partial t}+V^2\frac{\partial\rho}{\partial p}p'\frac{\partial p}{\partial t}=\rho V^3\frac{\partial p'}{\partial t'}/L+M^2p'\frac{\partial p}{\partial t}$$

であるから

$$\frac{\partial p}{\partial t}=pV^3\frac{\partial p'}{\partial t'}\Big/[L(1-M^2p')]$$

となる。同様にして

$$\frac{\partial p}{\partial x}=\rho V^2\frac{\partial p'}{\partial x'}\Big/[L(1-M^2p')],$$

$$\frac{\partial p}{\partial y}=\rho V^2\frac{\partial p'}{\partial y'}\Big/[L(1-M^2p')].$$

$$\frac{\partial p_b}{\partial x}=\rho_b V^2\frac{\partial p_b'}{\partial x}\Big/[L(1-M_b{}^2p_b)]$$

となる。

$$\frac{\partial\bar{p}}{\partial x}=\rho V^2\frac{\partial\bar{p}'}{\partial x'}\Big/L+V^2\frac{\partial\rho}{\partial p}\bar{p}'\left(\frac{\partial p_b}{\partial x}+\frac{\partial\bar{p}}{\partial x}\right)=\rho V^2\frac{\partial\bar{p}'}{\partial x'}\Big/L+M^2\rho_bV^2\bar{p}'\frac{\partial p_b'}{\partial x'}\Big/[L(1-M_b{}^2p_b')]+M^2\bar{p}'\frac{\partial\bar{p}}{\partial x}$$

であるから

$$\frac{\partial\bar{p}}{\partial x}=\rho V^2\left\{\frac{\partial\bar{p}'}{\partial x}+M^2(\rho_b/\rho)\bar{p}'\frac{\partial p_b'}{\partial x'}\Big/(1-M_b{}^2p_b')\right\}\Big/[L(1-M^2\bar{p}')]$$

となり

$$\frac{\partial p}{\partial x}=\rho V^2\left\{(\rho_b/\rho)\frac{\partial p_b'}{\partial x'}\Big/(1-M_b{}^2p_b')+\frac{\partial\bar{p}'}{\partial x'}\right\}\Big/[(1-M^2\bar{p}')]$$

となる。

これらを上の方程式を代入し零ディメンションの量を表わすダッシュを省略すれば

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-M^2\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/(1-M^2p),$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+M^2u\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/(1-M^2p)+\left\{(\rho_b/\rho)\frac{\partial p_b}{\partial x}\Big/(1-M_b{}^2p_b)+\frac{\partial\bar{p}}{\partial x}\right\}\Big/(1-M^2\bar{p})=F_x+\left[\mu_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-(\mu_1+\mu_2)\frac{\partial}{\partial x}\left\{M^2\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/\right.\right.$$

$$(1-M^2p)\Big\}+2\frac{\partial\mu_1}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\mu_1}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$-M^2\frac{\partial\mu_2}{\partial x}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/(1-M^2p)\Big]\Big/R,$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+M^2v\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/$$

$$(1-M^2p)+\frac{\partial p}{\partial y}\Big/(1-M^2)=F_y$$

$$+\Big[\mu_1\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-(\mu_1+\mu_2)\frac{\partial}{\partial y}$$

$$\Big\{M^2\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/(1-M^2p)\Big\}$$

$$+2\frac{\partial\mu_1}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial\mu_1}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$-M^2\frac{\partial\mu_2}{\partial y}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\Big/(1-M^2p)\Big]\Big/R$$

となる。

任意の形の物体に対してはこれに応ずる直交曲線座標で上の方程式を表わさなければいけない。こゝでは簡単のため上の方程式が利用出来る平板が迎角 0° で流体中を運動する場合を考える。層流の場合の境界層の厚さを $\bar{y}(x)$，乱流の場合の境界層の厚さを $\bar{y}(x,t)$ とする。y_1 は $\bar{y}(1)$ あるいは $\bar{y}(1,0)$ を表わすものとすればマッハ数 M は速さ V により y_1 のあるべき数に等しいものと置くことが出来る。u, v, p, μ_1, μ_2 を $x, \eta=y/\bar{y}$ の関数と考えこれを y_1 のべき級数に展開する。y_1^6 あるいは y_1^7 を省略し得る微小量とすれば有限項のべき級数となる。これを上の方程式に代入しマッハ数の場所による変化を考慮に入れないものとする。y_1^s の係数を等置すれば種々のマッハ数に対し次のような微分方程式が得られる。

3. $M=y_1^2$ の場合

$$u=u_0+y_1^2u_2+y_1^4u_4,\ v=y_1v_1+y_1^3v_3+y_1^5v_5,$$

$$p=y_1^2p_2+y_1^4p_4,\ \mu_1=\mu_{10}+y_1^2\mu_{12}+y_1^4\mu_{14},$$

$$\rho_b/\rho=1,\ \frac{\partial\bar{y}}{\partial t}=0,\ X=\frac{d\bar{y}}{dx}\Big/\bar{y}$$

とし y_1^6 の大さを省略すれば

$$\frac{dv_1}{d\eta}=\bar{y}\frac{\partial u_0}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{10}\frac{\partial u_0}{\partial\eta}\right)=-Ry_1\bar{y}\frac{\partial u_0}{\partial\eta}(\bar{y}u_0\eta X/y_1-v_1),$$

$$\frac{\partial p_2}{\partial\eta}=\left(\mu_{10}\frac{\partial^2 v_1}{\partial\eta^2}+2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_1}{\partial\eta}+\bar{y}\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta X/y_1\right)\Big/$$

$$(Ry_1\bar{y})+\frac{\partial v_1}{\partial\eta}(\bar{y}u_0\eta X/y_1-v_1),$$

$$\frac{\partial v_3}{\partial\eta}=\bar{y}\frac{\partial u_2}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{12}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big\{\bar{y}\frac{\partial(u_0u_2)}{\partial\eta}\eta X/y_1$$

$$-\left(v_1\frac{\partial u_2}{\partial\eta}+v_3\frac{\partial u_0}{\partial\eta}\right)\Big\}\Big/\bar{y}-\left(\frac{\partial p_{b2}}{\partial x}+\frac{\partial\bar{p}_2}{\partial\eta}\eta X\right)\Big]$$

$$-\Big[\mu_{10}\Big[\frac{\partial^2 u_2}{\partial\eta_2}+\bar{y}^2\Big\{\frac{\partial^2 u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\cdot$$

$$\left(X^2-\frac{dX}{dx}\right)\Big\}\Big/y_1^2\Big]+2\bar{y}^2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta^2X^2/y_1^2$$

$$+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big],$$

$$\frac{\partial p_4}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\Big\{\frac{\partial^2 v_1}{\partial\eta^2}\eta^2X^2+\frac{\partial v_1}{\partial\eta}\eta\left(X^2-\frac{dX}{dx}\right)\Big\}\Big/$$

$$y_1^2+\frac{\partial^2 v_3}{\partial\eta^2}\Big]+\mu_{12}\frac{\partial^2 v_1}{\partial\eta^2}+2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\right)$$

$$+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/(Ry_1\bar{y})$$

$$+\bar{y}\left(u_0\frac{\partial v_3}{\partial\eta}+u_2\frac{\partial v_1}{\partial\eta}\right)\eta X/y_1-\frac{\partial(v_1v_3)}{\partial\eta},$$

$$\frac{\partial v_5}{\partial\eta}=\bar{y}\frac{\partial u_4}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{14}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\left(\frac{\partial(u_0u_4)}{\partial\eta}+u_2\frac{\partial u_2}{\partial\eta}\right)\cdot$$

$$\eta X/y_1-\left(v_1\frac{\partial u_4}{\partial\eta}+v_3\frac{\partial u_2}{\partial\eta}+v_5\frac{\partial u_0}{\partial\eta}\right)\Big]\Big/\bar{y}$$

$$-\left(\frac{\partial p_{b4}}{\partial x}+\frac{\partial\bar{p}_4}{\partial\eta}\eta X\right)\Big]-\Big[\mu_{10}\Big[\frac{\partial^2 u_4}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2 u_2}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_2}{\partial\eta}\eta\left(X^2-\frac{dX}{dx}\right)\Big\}\Big/y_1^2\Big]+\mu_{12}\Big[\frac{\partial^2 u^2}{\partial\eta^2}$$

$$+\bar{y}^2\Big\{\frac{\partial^2 u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\left(X^2-\frac{dX}{dx}\right)\Big\}\Big/y_1^2\Big]$$

$$+2\bar{y}^2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_2}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\right)\eta^2X^2/y_1^2$$

$$+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_4}{\partial\eta}-\bar{y}\frac{\partial v_3}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big]$$

となる。

4. $M=y_1^{3/2}$ の場合

$$u=u_0+y_1^2u_2+y_1^4u_4,\ v=y_1v_1+y_1^3v_3+y_1^5v_5.$$

$$p=y_1^2p_2+y_1^4p_4,\ \mu_1=\mu_{10}+y_1^2\mu_{12}+y_1^4\mu_{14},$$

$$\rho_b/\rho=1+y_1^3r_3,\ \frac{\partial\bar{y}}{\partial t}=0$$

とすれば同様にして

$$\frac{\partial v_1}{\partial\eta}=\bar{y}\frac{\partial u_0}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{10}\frac{\partial u_0}{\partial\eta}\right)=-Ry_1\bar{y}\frac{\partial u_0}{\partial\eta}(\bar{y}u_0\eta X/y_1-v_1),$$

$$\frac{\partial p_2}{\partial \eta}=\Big(\mu_{10}\frac{\partial^2 v_1}{\partial\eta^2}+2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_1}{\partial\eta}+\bar{y}\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta X/y_1\Big)\Big/$$
$$(Ry_1\bar{y})+\frac{\partial v_1}{\partial\eta}(\bar{y}u_0\eta X/y_1-v_1),$$

$$\frac{\partial v_3}{\partial\eta}=\bar{y}\frac{\partial u_2}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\Big(\mu_{12}\frac{\partial u_0}{\partial\eta}\Big)=-R\bar{y}^2\Big[y_1\Big\{\bar{y}\frac{\partial(u_0u_2)}{\partial\eta}\eta X/y_1$$
$$-\Big(v_1\frac{\partial u_2}{\partial\eta}+v_3\frac{\partial u_0}{\partial\eta}\Big)\Big\}\Big/\bar{y}-\Big(\frac{\partial p_{b2}}{\partial x}+\frac{\partial\bar{p}_2}{\partial\eta}\eta X\Big)\Big]$$
$$-\Big[\mu_{10}\Big[\frac{\partial^2u_2}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\cdot$$
$$\Big(X^2-\frac{dX}{dx}\Big)\Big\}\Big/y_1^2\Big]+2\bar{y}^2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta^2X^2/y_1^2$$
$$+\frac{\partial\mu_{10}}{\partial\eta}\Big(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\partial X/y_1\Big)\Big],$$

$$\frac{\partial p_4}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\Big\{\frac{\partial^2v_1}{\partial\eta^2}\eta^2X^2+\frac{\partial v_1}{\partial\eta}\eta\Big(X^2-\frac{dX}{dx}\Big)\Big\}\Big/$$
$$y_1^2+\frac{\partial^2v_3}{\partial\eta^2}\Big]+\mu_{12}\frac{\partial^2v_1}{\partial\eta^2}$$
$$+2\Big(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\Big)+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\cdot$$
$$\Big(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial y}\eta X/y_1\Big)+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/$$
$$(Ry_1\bar{y})+\bar{y}\Big(u_0\frac{\partial v_3}{\partial\eta}+u_2\frac{\partial v_1}{\partial\eta}\Big)\eta X/y_1-\frac{\partial(v_1v_3)}{\partial\eta},$$

$$\frac{\partial v_5}{\partial\eta}=\bar{y}\frac{\partial u_4}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\Big(\mu_{14}\frac{\partial u_0}{\partial\eta}\Big)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big(\frac{\partial(u_0u_4)}{\partial\eta}+u_2\frac{\partial u_2}{\partial\eta}\Big)\cdot$$
$$\eta X/y_1-\Big(v_1\frac{\partial u_4}{\partial\eta}+v_3\frac{\partial u_2}{\partial\eta}+v_5\frac{\partial u_0}{\partial\eta}\Big)\Big]\Big/\bar{y}$$
$$-\Big(\frac{\partial p_{b4}}{\partial x}+\frac{\partial\bar{p}_4}{\partial\eta}\eta X\Big)\Big]-\Big[\mu_{10}\Big[\frac{\partial^2u_4}{\partial\eta^2}$$
$$+\bar{y}^2\Big\{\frac{\partial^2u_2}{\partial\eta^2}\eta^2X^2+\frac{\partial u_2}{\partial\eta}\eta\Big(X^2-\frac{dX}{dx}\Big)\Big\}\Big/y_1^2\Big]$$
$$+\mu_{12}\Big[\frac{\partial^2u_2}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\cdot$$
$$\Big(X^2-\frac{dX}{dx}\Big)\Big\}\Big/y_1^2\Big]$$
$$+2\bar{y}^2\Big(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_2}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big)\eta^2X^2/y_1^2$$
$$+\frac{\partial\mu_{10}}{\partial\eta}\Big(\frac{\partial u_4}{\partial\eta}-\bar{y}\frac{\partial v_3}{\partial\eta}\eta X/y_1\Big)$$
$$+\frac{\partial\mu_{12}}{\partial\eta}\Big(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\Big)\Big],$$

$$r_3=-y_1u_0\frac{\partial p_2}{\partial\eta}(\bar{y}u_0\eta X/y_1-v_1)\Big/\Big(\bar{y}\frac{\partial p_{b2}}{\partial x}\Big)$$

となる。

5. $M=y_1$ の場合

$$u=u_0+y_1^2u_2+y_1^4u_4,\quad v=y_1v_1+y_1^3v_3+y_1^5v_5,$$
$$p=y_1^2p_2+y_1^4p_4+y_1^6p_6,\quad \mu_1=\mu_{10}+y_1^2\mu_{12}+y_1^4\mu_{14}$$
$$+y_1^6\mu_{16},\quad \rho_b/\rho=1+y_1^2r_2+y_1^4r_4,$$
$$T=\frac{\partial\bar{y}}{\partial t}\Big/\bar{y},\quad X=\frac{\partial\bar{y}}{dx}\Big/\bar{y}$$

とし y_1^7 の大きさを省略すれば

$$\frac{\partial v_1}{\partial\eta}=\bar{y}\frac{\partial u_0}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\Big(\mu_{10}\frac{\partial u_0}{\partial\eta}\Big)=-Ry_1\bar{y}\frac{\partial u_0}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\},$$

$$\frac{\partial p_2}{\partial\eta}=\Big(\mu_{10}\frac{\partial^2v_1}{\partial\eta^2}+2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_1}{\partial\eta}+\bar{y}\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta X/y_1\Big)\Big/$$
$$(Ry_1\bar{y})+\frac{\partial v_1}{\partial\eta}\{y\eta(T+u_0X)/y_1-v_1\},$$

$$\frac{\partial v_3}{\partial\eta}=\bar{y}\frac{\partial u_2}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\Big(\mu_{12}\frac{\partial u_0}{\partial\eta}\Big)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big\{\frac{\partial u_2}{\partial\eta}\eta(T+u_0X)$$
$$+u_2\frac{\partial u_0}{\partial\eta}\eta X\Big\}\Big/y_1-\Big(v_1\frac{\partial u_2}{\partial\eta}+v_3\frac{\partial u_0}{\partial\eta}\Big)\Big]\Big/\bar{y}$$
$$-\Big(\frac{\partial p_{b2}}{\partial x}+\frac{\partial\bar{p}_2}{\partial\eta}\eta X\Big)\Big]-\Big[\mu_{10}\Big[\frac{\partial^2u_2}{\partial\eta^2}$$
$$+\bar{y}^2\Big\{\frac{\partial^2u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/y_1^2\Big]$$
$$+2\bar{y}^2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta^2X^2/y_1^2$$
$$+\frac{\partial\mu_{10}}{\partial\eta}\Big(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\Big)\Big],$$

$$\frac{\partial p_4}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\Big\{\frac{\partial^2v_1}{\partial\eta^2}\eta^2X^2+\frac{\partial v_1}{\partial\eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/y_1^2$$
$$+\frac{\partial^2v_3}{\partial\eta^2}\Big]+\mu_{12}\frac{\partial^2v_1}{\partial\eta^2}+2\Big(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\Big)$$
$$+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\Big(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\Big)$$
$$+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/(Ry_1\bar{y})+\bar{y}\Big\{\frac{\partial v_3}{\partial\eta}\eta(T+u_0X)$$
$$+u_2\frac{\partial v_1}{\partial\eta}\eta\times\Big\}\Big/y_1-\frac{\partial(v_1v_3)}{\partial\eta},$$

$$\frac{\partial v_5}{\partial\eta}=\bar{y}\Big\{\frac{\partial u_4}{\partial\eta}\eta X+\frac{\partial p_2}{\partial\eta}\eta(T+u_0X)\Big\}\Big/y_1-v_1\frac{\partial u_2}{\partial\eta},$$

$$\frac{\partial}{\partial\eta}\Big(\mu_{14}\frac{\partial u_0}{\partial\eta}\Big)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big\{\frac{\partial u_4}{\partial\eta}\eta(T+u_0X)$$
$$+\Big(u_2\frac{\partial u_2}{\partial\eta}+u_4\frac{\partial u_0}{\partial\eta}\Big)\eta X\Big\}\Big/y_1-\Big(v_1\frac{\partial u_4}{\partial\eta}+v_3\frac{\partial u_2}{\partial\eta}$$
$$+v_5\frac{\partial u_0}{\partial\eta}\Big)\Big]\Big/\bar{y}-\Big(\frac{\partial p_{b4}}{\partial x}+\frac{\partial\bar{p}_4}{\partial\eta}\eta X\Big)\Big]$$
$$-\Big[\mu_{10}\Big[\frac{\partial^2u_4}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2u_2}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_2}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2\Big]+\mu_{12}\Big[\frac{\partial^2 u_2}{\partial\eta^2}$$

$$+\bar{y}^2\left\{\frac{\partial^2 u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\right\}\Big/y_1^2\Big]$$

$$+2\bar{y}^2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_2}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\right)\eta^2X^2/y_1^2$$

$$+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_4}{\partial\eta}-\bar{y}\frac{\partial v_3}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big)\Big],$$

$$r_2=-y_1u_0\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big/\left(\bar{y}\frac{\partial p_{b2}}{\partial x}\right)$$

$$\frac{\partial p_6}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\left\{\frac{\partial^2 v_3}{\partial\eta^2}\eta^2X^2+\frac{\partial X}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\right\}\Big/y_1^2$$

$$+\frac{\partial^2 v_5}{\partial\eta^2}\Big]+\mu_{12}\Big[\bar{y}^2\Big\{\frac{\partial^2 v_1}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial v_1}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2+\frac{\partial^2 v_3}{\partial\eta^2}\Big]+\mu_{14}\frac{\partial^2 v_1}{\partial\eta^2}$$

$$+2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_5}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_5}{\partial\eta}+\frac{\partial\mu_{14}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\right)$$

$$+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_4}{\partial\eta}-\bar{y}\frac{\partial v_3}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-y\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{14}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/(Ry_1\bar{y})$$

$$+\bar{y}\left\{\frac{\partial v_5}{\partial\eta}\eta(T+u_0X)+\left(u_2\frac{\partial v_3}{\partial\eta}+u_4\frac{\partial v_1}{\partial\eta}\right)\eta X\right\}\Big/$$

$$y_1-\frac{\partial(v_1v_5)}{\partial\eta}-v_3\frac{\partial v_3}{\partial\eta}-p_2\frac{\partial p_2}{\partial\eta},$$

$$\frac{\partial}{\partial\eta}\left[\mu_2\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\right]$$

$$=-Ry_1\bar{y}v_1\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}$$

$$-\mu_{10}\frac{\partial}{\partial\eta}\left[\frac{\partial p_2}{\partial\eta}\left\{\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\right]\right.,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{16}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big\{\bar{y}\frac{\partial(u_2u_4)}{\partial\eta}\eta X/y_1$$

$$-\left(v_3\frac{\partial u_4}{\partial\eta}+v_5\frac{\partial u_2}{\partial\eta}\right)\Big\}\Big/\bar{y}-\frac{\partial p_{b6}}{\partial x}-p_2\frac{\partial p_{b2}}{\partial x}$$

$$-\left(\frac{\partial p_6}{\partial\eta}+p_2\frac{\partial p_2}{\partial\eta}\right)\eta X\Big]-\Big[\mu_{10}\bar{y}^2\Big\{\frac{\partial^2 x_4}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_4}{\partial\eta}\eta\left(X^2-\frac{dX}{\partial x}\right)\Big\}\Big/y_1^2+\mu_{12}\Big[\frac{\partial^2 u_4}{\partial\eta^2}$$

$$+\bar{y}^2\left\{\frac{\partial^2 u_2}{\partial\eta^2}\eta^2X^2+\frac{\partial u_2}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\right\}\Big/y_1^2\Big]$$

$$+\mu_{14}\Big[\frac{\partial^2 u_2}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2 u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\cdot$$

$$\left(X^2-\frac{\partial X}{dx}\right)\Big\}\Big/y_1^2\Big]+2\bar{y}^2\Big(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_4}{\partial\eta}$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_4}{\partial\eta}+\frac{\partial\mu_{14}}{\partial\eta}\Big)\eta^2X^2/y_1^2$$

$$-\bar{y}\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_5}{\partial\eta}\eta X/y_1+\frac{\partial\mu_{12}}{\partial\eta}\left(\frac{\partial u_4}{\partial\eta}-\bar{y}\frac{\partial v_3}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{14}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-y\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big)\Big]$$

$$r_4=-\Big[y_1\Big[u_0\Big[\bar{y}\left\{\frac{\partial p_4}{\partial\eta}(T+u_0X)+u_2\frac{\partial p_2}{\partial\eta}\eta X\right\}\Big/y_1$$

$$-\left(v_1\frac{\partial p_4}{\partial\eta}+v_3\frac{\partial p_2}{\partial\eta}\right)\Big]+u_2\frac{\partial p_2}{\partial\eta}\{\eta\bar{y}(T+u_0X)/y_1$$

$$-v_1\}\Big]\Big/\bar{y}+r_2\frac{\partial p_{b4}}{\partial x}+\Big[\bar{y}^2\Big[\Big[\Big\{\mu_{10}\left(\frac{\partial^2 p_2}{\partial\eta^2}\eta+\frac{\partial p_2}{\partial\eta}\right)$$

$$+\frac{\partial}{\partial\eta}\left(\mu_{20}\frac{\partial p_2}{\partial\eta}\right)\eta+\mu_{20}\frac{\partial p_2}{\partial\eta}\Big\}(T+u_0X)$$

$$+\left(\mu_{10}\frac{\partial u_0}{\partial\eta}+\frac{\partial(\mu_2{}^0u_0)}{\partial\eta}\right)\frac{\partial p_2}{\partial\eta}\eta X\Big]\eta X$$

$$-(\mu_{10}+\mu_{20})\frac{\partial p_2}{\partial\eta}\eta\left(\frac{\partial T}{\partial x}+u_0\frac{\eta X}{\partial x}\right)\Big]\Big/y_1^2$$

$$-\bar{y}\Big[\left\{\mu_{10}\frac{\partial}{\partial\eta}\left(v_1\frac{\partial p_2}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\mu_{20}v_1\frac{\partial p_2}{\partial\eta}\right)\right\}\eta$$

$$+(\mu_{10}+\mu_{20})v_1\frac{\partial p_2}{\partial\eta}\Big]X/y_1\Big]\Big/(R\bar{y}^2)\Big]\Big/\frac{\partial p_{b2}}{\partial x}$$

となる。

6. $M=y_1^{1/2}$ の場合

$u=u_0+y_1^2u_2+y_1^4u_4+y_1^5u_5$,

$v=y_1v_1+y_1^3v_3+y_1^4v_4+y_1^5v_5+y_1^6v_6$,

$p=y_1^2p_2+y_1^4p_4+y_1^5p_5+y_1^6p_6$,

$\mu_1=\mu_{10}+y_1^2\mu_{12}+y_1^3\mu_{13}+y_1^4\mu_{14}+y_1^5\mu_{15}+y_1^6\mu_{16}$,

$\mu_2=\mu_{20}+y_1^2\mu_{22}$, $\rho_b/\rho=1+y_1r_1+y_1^3r_3+y_1^4r_4$

とすれば同様にして

$$\frac{\partial v_1}{\partial\eta}=\bar{y}\frac{\partial u_0}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{10}\frac{\partial u_0}{\partial\eta}\right)=-Ry_1\bar{y}\frac{\partial u_0}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\},$$

$$\frac{\partial p_2}{\partial\eta}=\Big(\mu_{10}\frac{\partial^2 v_1}{\partial\eta^2}+2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_1}{\partial\eta}$$

$$+\bar{y}\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta X/y_1\Big)\Big/(Ry_1\bar{y})$$

$$+\frac{\partial v_1}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\},$$

$$\frac{\partial v_3}{\partial\eta}=\bar{y}\frac{\partial u^2}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{12}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big\{\frac{\partial u_2}{\partial\eta}\eta(T+u_0X)$$

$$+u_2\frac{\partial u_0}{\partial\eta}\eta X\Big\}\Big/y_1-\left(v_1\frac{\partial u_2}{\partial\eta}+v_3\frac{\partial u_0}{\partial\eta}\right)\Big]\Big/\bar{y}$$

$$-\frac{\partial p_{b2}}{\partial x}-\frac{\partial\bar{p}_2}{\partial\eta}\eta X\Big]-\Big[\mu_{10}\Big[\frac{\partial^2 u_2}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2 u_0}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_0}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2\Big]$$

$$+2\bar{y}^2\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta^2X^2/y_1^2$$

$$+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_0}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big)\Big],$$

$$\frac{\partial p_4}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\Big\{\frac{\partial^2v_1}{\partial\eta^2}\eta^2X^2+\frac{\partial v_1}{\partial\eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/y_1^2$$

$$+\frac{\partial^2v_1}{\partial\eta^2}\Big]+\mu_{12}\frac{\partial^2v_1}{\partial\eta^2}+2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\right)$$

$$+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/(Ry_1\bar{y})$$

$$+\bar{y}\Big\{\frac{\partial v_3}{\partial\eta}\eta(T+u_0X)+u_2\frac{\partial v_1}{\partial\eta}\eta X\Big\}\Big/y_1-\frac{\partial(v_1v_3)}{\partial\eta},$$

$$\frac{\partial v_4}{\partial\eta}=\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\},$$

$$\frac{\partial}{\partial\eta}\left(\mu_{13}\frac{\partial u_0}{\partial\eta}\right)=Ry_1\bar{y}v_3\frac{\partial u_0}{\partial\eta},$$

$$r_1=-y_1u_0\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big/\left(\bar{y}\frac{\partial p_{b2}}{\partial x}\right),$$

$$\frac{\partial p_5}{\partial\eta}=\Big\{\mu_{10}\frac{\partial^2v_4}{\partial\eta^2}+\mu_{13}\frac{\partial^2v_1}{\partial\eta^2}$$

$$+2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{13}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\right)$$

$$+\bar{y}\frac{\partial\mu_{13}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\eta X/y_1\Big\}\Big/(Ry_1\bar{y})$$

$$+\bar{y}\frac{\partial v_4}{\partial\eta}\eta(T+u_0X)/y_1-\frac{\partial(v_1v_4)}{\partial\eta}-p_2\frac{\partial p_2}{\partial\eta},$$

$$\frac{\partial}{\partial\eta}\Big[\mu_{20}\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big]$$

$$=-Ry_1\bar{y}v_1\frac{\partial p_2}{\partial\eta}\{y\eta(T+u_0X)/y_1-v_1\}$$

$$-\mu_{10}\frac{\partial}{\partial\eta}\Big[\frac{\partial p_2}{\partial\eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big],$$

$$\frac{\partial v_5}{\partial\eta}=\bar{y}\frac{\partial u_4}{\partial\eta}\eta X/y_1,$$

$$\frac{\partial}{\partial\eta}\left(\mu_{14}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big\{\frac{\partial u_4}{\partial\eta}\eta(T+u_0X)$$

$$+\left(u_2\frac{\partial u_2}{\partial\eta}+u_4\frac{\partial u_0}{\partial\eta}\right)\eta X\Big\}\Big/y_1$$

$$-\left(v_1\frac{\partial u_4}{\partial\eta}+v_3\frac{\partial u_2}{\partial\eta}+v_5\frac{\partial u_0}{\partial\eta}\right)\Big]\Big/\bar{y}$$

$$-\left(\frac{\partial p_{b4}}{\partial x}+\frac{\partial p_4}{\partial\eta}\eta\times\right)\Big]-\Big[\mu_{10}\Big[\frac{\partial^2u_4}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2u_4}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_2}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2\Big]+\mu_{12}\Big[\frac{\partial^2u_2}{\partial\eta^2}$$

$$+\bar{y}^2\Big\{\frac{\partial^2u_0}{\partial\eta^2}\eta^2X^2+\frac{\partial u_0}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2\Big]$$

$$+2\bar{y}^2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial u_2}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\right)\eta^2X^2/y_1^2$$

$$+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_3}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big],$$

$$\frac{\partial p_6}{\partial\eta}=\Big[\mu_{10}\Big[\bar{y}^2\Big\{\frac{\partial^2v_3}{\partial\eta^2}\eta^2X^2+\frac{\partial v_3}{\partial\eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/$$

$$y_1^2+\frac{\partial^2v_5}{\partial\eta^2}\Big]+\mu_{12}\Big[\bar{y}^2\Big\{\frac{\partial^2v_1}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial v_1}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2+\frac{\partial^2v_3}{\partial\eta^2}\Big]+\mu_{13}\frac{\partial^2v_1}{\partial\eta^2}$$

$$+2\left(\frac{\partial\mu_{10}}{\partial\eta}\frac{\partial v_5}{\partial\eta}+\frac{\partial\mu_{12}}{\partial\eta}\frac{\partial v_3}{\partial\eta}+\frac{\partial\mu_{14}}{\partial\eta}\frac{\partial v_1}{\partial\eta}\right)$$

$$+\bar{y}\Big\{\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_4}{\partial\eta}-y\frac{\partial v_3}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{12}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-\bar{y}\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{14}}{\partial\eta}\frac{\partial u_0}{\partial\eta}\Big\}\eta X/y_1\Big]\Big/(Ry_1\bar{y})$$

$$+\bar{y}\Big\{\frac{\partial v_5}{\partial\eta}\eta(T+u_0X)+\left(u_2\frac{\partial v_3}{\partial\partial}+u_4\frac{\partial v_1}{\partial\eta}\right)\eta X\Big\}\Big/$$

$$y_1-\left(\frac{\partial(v_1v_5)}{\partial\eta}+v_3\frac{\partial v_3}{\partial\eta}\right),$$

$$\frac{\partial v_6}{\partial\eta}=\bar{y}\Big\{\frac{\partial u_5}{\partial\eta}\eta X+\frac{\partial p_4}{\partial\eta}\eta(T+u_0X)+u_2\frac{\partial p_2}{\partial\eta}\eta X\Big\}\Big/y_1$$

$$-\left(v_1\frac{\partial p_4}{\partial\eta}+v_3\frac{\partial y_2}{\partial\eta}\right),$$

$$\frac{\partial}{\partial\eta}\left(\mu_{15}\frac{\partial u_0}{\partial\eta}\right)=-R\bar{y}^2\Big[y_1\Big[\bar{y}\Big\{\frac{\partial u_5}{\partial\eta}\eta(T+u_0X)$$

$$+u_5\frac{\partial u_0}{\partial\eta}\eta X\Big\}\Big/y_1-\Big(v_1\frac{\partial u_5}{\partial\eta}+v_3\frac{\partial u_2}{\partial\eta}$$

$$+v_6\frac{\partial u_0}{\partial\eta}\Big)\Big]\Big/\bar{y}-\left(\frac{\partial p_{b5}}{\partial x}+p_2\frac{\partial p_{b2}}{\partial x}\right)$$

$$-\left(\frac{\partial p_5}{\partial\eta}+\bar{p}_2\frac{\partial p_2}{\partial\eta}\right)\eta X\Big]-\Big[\mu_{10}\frac{\partial^2u_5}{\partial\eta^2}$$

$$+\mu_{13}\Big[\frac{\partial^2u_2}{\partial\eta^2}+\bar{y}^2\Big\{\frac{\partial^2u_0}{\partial\eta^2}\eta^2X^2$$

$$+\frac{\partial u_0}{\partial\eta}\eta\left(X^2-\frac{\partial X}{\partial x}\right)\Big\}\Big/y_1^2\Big]+2\bar{y}^2\frac{\partial\mu_{13}}{\partial\eta}\cdot$$

$$\frac{\partial u_0}{\partial\eta}\eta^2X^2/y_1^2+\frac{\partial\mu_{10}}{\partial\eta}\left(\frac{\partial u_5}{\partial\eta}-\bar{y}\frac{\partial v_4}{\partial\eta}\eta X/y_1\right)$$

$$+\frac{\partial\mu_{13}}{\partial\eta}\left(\frac{\partial u_2}{\partial\eta}-y\frac{\partial v_1}{\partial\eta}\eta X/y_1\right)\Big)\Big]$$

$$r_3=-\Big\{y_1\Big[u_0\Big[\bar{y}\Big\{\frac{\partial p_4}{\partial\eta}\eta(T+u_0X)+u_2\frac{\partial\eta_2}{\partial\eta}\eta X\Big\}\Big/y_1$$

$$-\left(v_1\frac{\partial p_4}{\partial\eta}+v_3\frac{\partial p_2}{\partial\eta}\right)\Big]+u_2\frac{\partial p_2}{\partial\eta}\{y\eta(T+u_0X)/y_1$$

$$-v_1\}\Big]\Big/\bar{y}+r_1\frac{\partial p_{b2}}{\partial x}+\Big[\bar{y}^2\Big[\Big[\Big\{\mu_{10}\left(\frac{\partial^2p_2}{\partial\eta^2}\eta+\frac{\partial p_2}{\partial\eta}\right)$$

$$+\frac{\partial}{\partial\eta}\left(\mu_{20}\frac{\partial p_2}{\partial\eta}\right)\eta+\mu_{20}\frac{\partial p_2}{\partial\eta}\Big\}(T+u_0X)$$

$$+\left(\mu_{10}\frac{\partial u_0}{\partial\eta}+\frac{\partial(\mu_{20}u_0)}{\partial\eta}\right)\frac{\partial p_2}{\partial\eta}\eta X\Big]\eta X$$

$$-(\mu_{10}+\mu_{20})\frac{\partial p_2}{\partial\eta}\eta\left(\frac{\partial T}{\partial x}+u_0\frac{\partial X}{\partial x}\right)\Big]\Big/y_1^2$$

$$-\bar{y}\Big[\Big\{\mu_{10}\frac{\partial}{\partial\eta}\left(v_1\frac{\partial p_2}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\mu_{20}v_1\frac{\partial p_2}{\partial\eta}\right)\Big\}\eta$$

$$+(\mu_{10}+\mu_{20})v_1\frac{\partial p_2}{\partial \eta}\Big]Xy_1\Big]\Big/(Ry^2)\Big\}\Big/\frac{\partial p_{b2}}{\partial x},$$

$$\frac{\partial}{\partial \eta}\Big[\mu_{22}\frac{\partial p_2}{\partial \eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big]$$

$$=-Ry_1\bar{y}\Big[v_1\Big[\bar{y}\Big\{\frac{\partial p_4}{\partial \eta}\eta(T+u_0X)$$

$$+u_2\frac{\partial p_2}{\partial \eta}\eta X\Big\}\Big/y_1-\Big(v_1\frac{\partial p_4}{\partial \eta}+v_3\frac{\partial p_2}{\partial \eta}\Big)\Big]$$

$$+v_3\frac{\partial p_2}{\partial \eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big]$$

$$-\frac{\partial}{\partial \eta}\Big[\mu_{20}\Big[\bar{y}\Big\{\frac{\partial p_4}{\partial \eta}\eta(T+u_0X)+u_2\frac{\partial p_2}{\partial \eta}\eta X\Big\}\Big/y_1$$

$$-\Big(v_1\frac{\partial p_4}{\partial \eta}+v_3\frac{\partial p_2}{\partial \eta}\Big)\Big]\Big]$$

$$-\mu_{10}\frac{\partial}{\partial \eta}\Big[\bar{y}\Big\{\frac{\partial p_4}{\partial \eta}\eta(T+u_0X)+u_2\frac{\partial p_2}{\partial \eta}\eta X\Big\}\Big/y_1$$

$$-\Big(v_1\frac{\partial p_4}{\partial \eta}+v_3\frac{\partial p_2}{\partial \eta}\Big)\Big]$$

$$-\mu_{12}\frac{\partial}{\partial \eta}\Big[\frac{\partial p_2}{\partial \eta}\{\bar{y}\eta(T+u_0X)/y_1-v_1\}\Big],$$

$$\frac{\partial}{\partial \eta}\Big(\mu_{16}\frac{\partial u_0}{\partial \eta}\Big)=-R\bar{y}^2\Big[y_1\Big\{\bar{y}\frac{\partial(u_2u_4)}{\partial \eta}\eta X/y_1$$

$$-\Big(v_3\frac{\partial u_4}{\partial \eta}+v_5\frac{\partial u_2}{\partial \eta}\Big)\Big\}\Big/\bar{y}-\Big(\frac{\partial p_{b2}}{\partial x}+\frac{\partial p_6}{\partial \eta}\eta X\Big)\Big]$$

$$-\Big[\mu_{10}\bar{y}^2\Big\{\frac{\partial^2 u_4}{\partial \eta^2}\eta^2X^2+\frac{\partial u_4}{\partial \eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/{y_1}^2$$

$$+\mu_{12}\Big[\frac{\partial^2 u_4}{\partial \eta^2}+\bar{y}^2\Big\{\frac{\partial^2 u_2}{\partial \eta^2}\eta^2X^2$$

$$+\frac{\partial u_2}{\partial \eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/{y_1}^2\Big]+\mu_{14}\Big[\frac{\partial^2 u_2}{\partial \eta^2}$$

$$+\bar{y}^2\Big\{\frac{\partial^2 u_0}{\partial \eta^2}\eta^2X^2+\frac{\partial u_0}{\partial \eta}\eta\Big(X^2-\frac{\partial X}{\partial x}\Big)\Big\}\Big/{y_1}^2\Big]$$

$$+2\bar{y}^2\Big(\frac{\partial \mu_{10}}{\partial \eta}\frac{\partial u_4}{\partial \eta}+\frac{\partial \mu_{12}}{\partial \eta}\frac{\partial u_2}{\partial \eta}$$

$$+\frac{\partial \mu_{14}}{\partial \eta}\frac{\partial u_0}{\partial \eta}\Big)\eta^2X^2/{y_1}^2-\bar{y}\frac{\partial \mu_{10}}{\partial \eta}\frac{\partial v_5}{\partial \eta}\eta X/y_1$$

$$+\frac{\partial \mu_{12}}{\partial \eta}\Big(\frac{\partial u_4}{\partial \eta}-\bar{y}\frac{\partial v_3}{\partial \eta}\eta X/y_1\Big)$$

$$+\frac{\partial \mu_{14}}{\partial \eta}\Big(\frac{\partial u_2}{\partial \eta}-\bar{y}\frac{\partial v_1}{\partial \eta}\eta X/y_1\Big)\Big]$$

$$r_4=-\Big[y_1u_0\Big\{\bar{y}\frac{\partial p_5}{\partial \eta}\eta(T+u_0X)/y_1$$

$$-\Big(v_1\frac{\partial p_5}{\partial \eta}+v_3\frac{\partial p_2}{\partial \eta}\Big)\Big\}\Big/\bar{y}+r_2\frac{\partial p_{b3}}{\partial x}\Big]\Big/\frac{\partial p_{b2}}{\partial x}$$

となる。

u あるいは v が与えられるときは上の微分方程式により v あるいは u p, μ_1, μ_2 を求めることが出来る。その詳細については次報で報告する。

7. 結果の考察

上に得られた結果および過去に出された報告から次のような事柄が分る。

1. $M\leqq {y_1}^2$ の場合には圧縮性の影響を考える必要がない。$M={y_1}^{3/2}$ の場合には ${y_1}^3$ の大さで，$M=y_1$ の場合には ${y_1}^2$ の大さで，$M={y_1}^{1/2}$ の場合には y_1 の大さで圧縮性の影響が表われる。

2. この圧縮性の影響により $M=y_1$ の場合には境界層の厚さは時間的に変化するようになる。$M\leqq {y_1}^2$ の場合には境界層の厚さの変動を起す原動力がない。$M={y_1}^{3/2}$ の場合およびその附近では単に圧縮されるだけで境界層の厚さの変動の原動力とならない場合と圧縮が変動の原動力となる場合とが考えられる。故に $M=y_1$ の場合流は乱流となり，$M\leqq {y_1}^2$ の場合には層流である。${y_1}^2<M<y_1$ の範囲は不安定領域と考えるべきであることが推測出来る。

3. 第2粘性係数については今までなにか不分明のものがあつたが上の式でその意義が明かになりその決定法が得られた。

4. 応力と歪率との関係の一般化によつて得られる μ_{10} の値は明治大学科学技術研究所紀要[3] の中に与えられている。その結果によればニュートン流体の場合のように境界層内全部にわたり $\mu_1=1$ と見做すのは無理であることを示している。近似的に $\mu_1=1$ と置き得るのは $M\leqq {y_1}^2$ の場合，即ち物体の運動速度が非常に小さいときで境界層の約 1/2 の範囲に限られる。テイラーの回転同心円柱内の流の場合もこの中に含まれるものと考えてよい。

5. $\mu_1=\mu_2=0$ の場合即ち理想流体の流の物体表面における流を粘性流体の境界層境界上の第1近似の流にとる。これから多項式近似による境界層の速度分布の第1近似を求める。これを利用して v, μ_{10} を求める。

次に境界層外の流に対する第1次近似としては v_1 を $1/\eta$ の関数として与えこれから u_0, μ_{10}, p_2 を求める。

これらの値を利用して境界層の境界上におけるベルヌーイの方程式から u_2 を求める。

上に与えられた方程式を利用して以下同様のことを繰返せば高次の近似値および ρ_b/ρ の値を求めることが出来る。

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